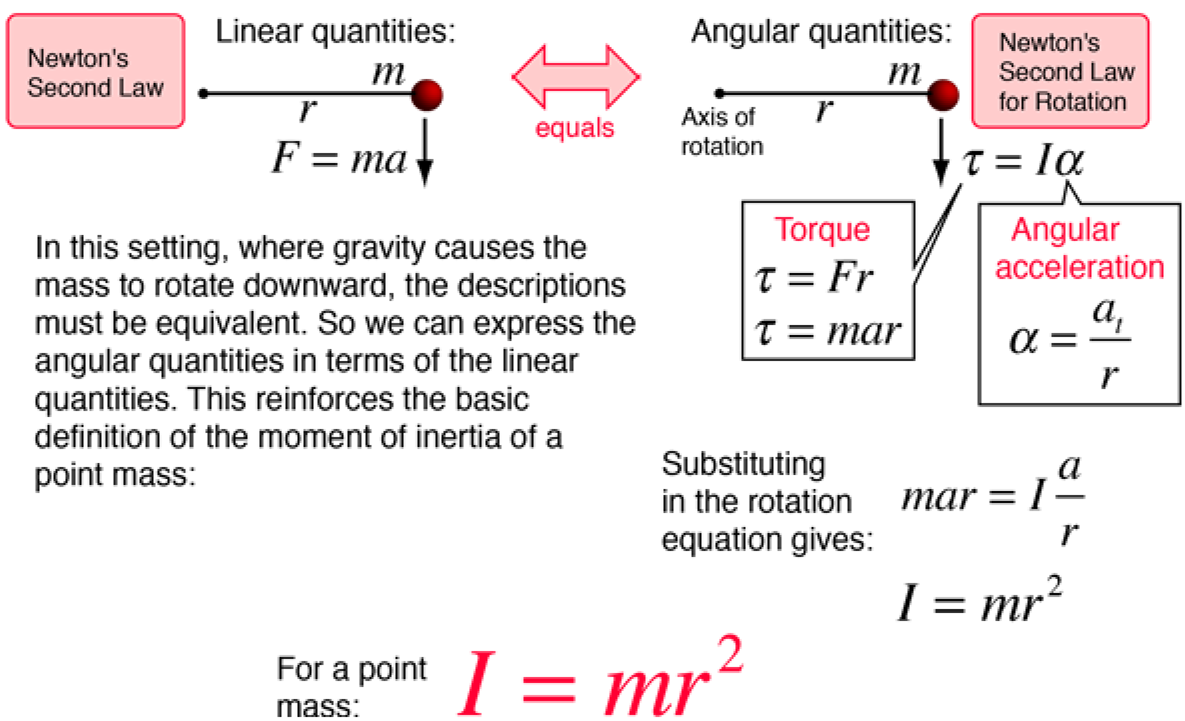
Rotational Inertia

Rotational inertia plays a similar role in rotational mechanics to mass in linear mechanics. Indeed, the rotational inertia of an object depends on its mass. It also depends on the distribution of that mass relative to the axis of rotation.

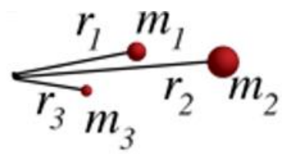
	Rotational Inertia / Moment of Inertia ***I***
Definition	Rotational inertia is a property of any object which can be rotated. It tells us how difficult it is to change the rotational velocity of the object around a given rotational axis.
Magnitude	$\boldsymbol{I \propto MR^2}$ (Rotational Inertia is proportional with mass *M* and rotate radius *R*)
Vector / Scalar	
Unit	$kg \cdot m^2$

1. Rotational and Linear Example

A mass m is placed on a rod of length r and negligible mass and constrained to rotate about a fixed axis. If the mass is released from a horizontal orientation, it can be described either in terms of force and acceleration with Newton's second law for linear motion, or as a pure rotation about the axis with Newton's second law for rotation. This provides a setting for comparing linear and rotational quantities for the same system. This process leads to the expression for the moment of inertia of a point mass.

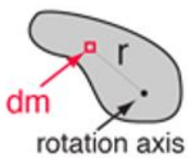


2. List rotational inertia of different shapes



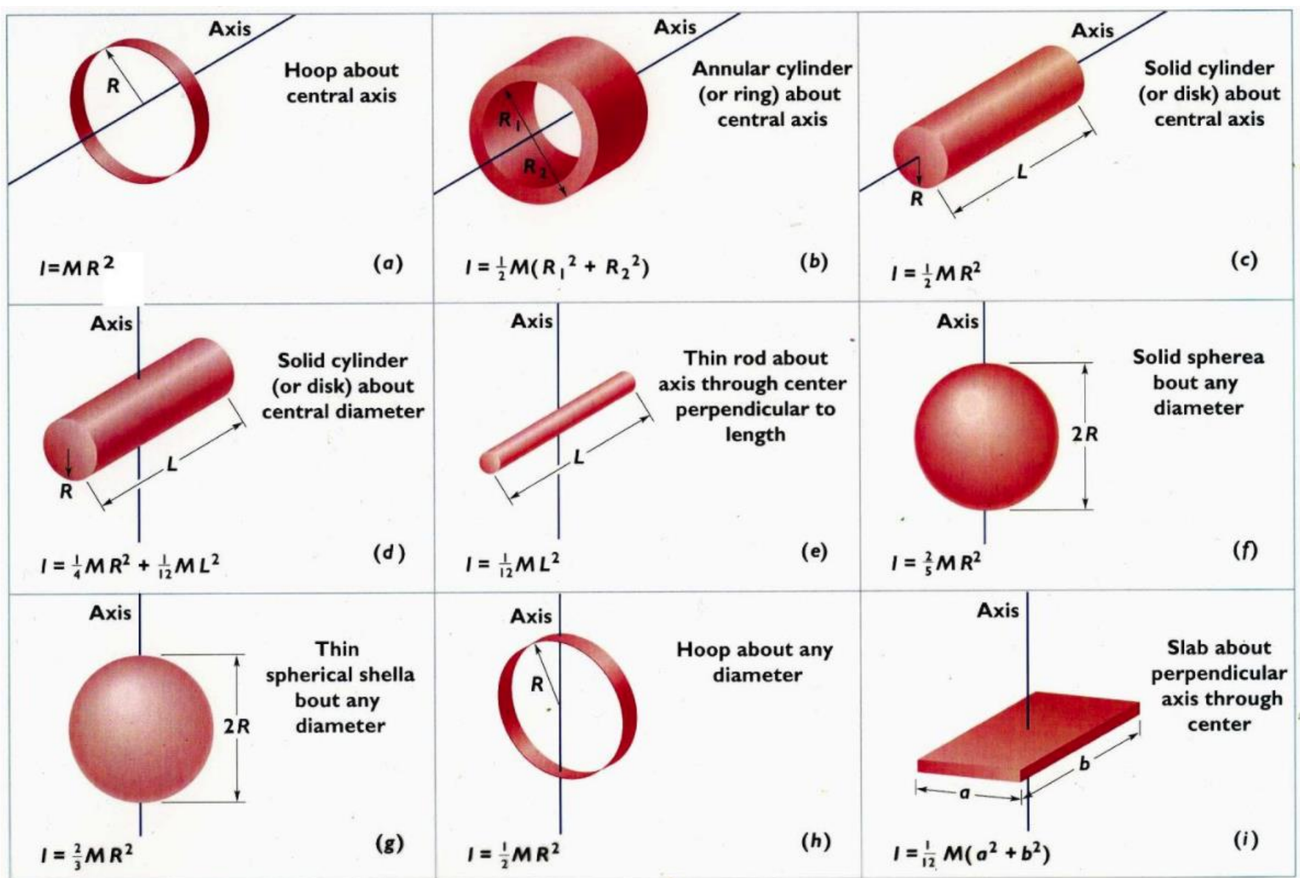
$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

Sum of the point mass moments of inertia.



$$I = \int_0^M r^2 dm$$

Continuous mass distributions require an infinite sum of all the point mass moments which make up the whole. This is accomplished by an integration over all the mass.



3. Rotational-Linear Parallels

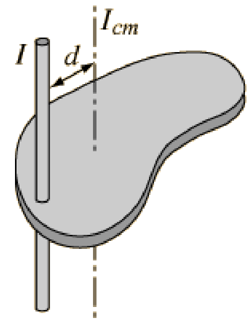
	Linear Motion	Rotational Motion	Supplement
Displacement			
Velocity			
Acceleration			
Kinematic Equations			
Inertia			
Newton's 2nd Law			
Work			
Kinetic Energy			
Work-KE Theorem			
Mechanical Energy			
Power			
Impulse			
Momentum			
Impulse-Momentum Theorem			
Momentum Conservation			

	Linear Motion	Rotational Motion	Supplement
Displacement	S	θ	$S = \theta \cdot r$
Velocity	v	ω	$v = \omega \cdot r$
Acceleration	a_t	α	$a_t = \alpha \cdot r$
Kinematic Equations	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	Constant Acceleration
	$S = v_0 t + \frac{1}{2} at^2$	$\theta = \omega_0 t + \frac{1}{2} \alpha t^2$	
	$v_t^2 - v_0^2 = 2aS$	$\omega_t^2 - \omega_0^2 = 2\alpha\theta$	
	$S = \frac{1}{2}(v_0 + v_t)\, t$	$\theta = \frac{1}{2}(\omega_0 + \omega_t)\, t$	
Inertia	m	I	$I \propto mr^2$
Newton's 2nd Law	$F = ma$	$\tau = I\alpha$	
Work	$W_L = F \cdot S_{//}$	$W_R = \tau \cdot \theta$	$_{L:}$ *Linear*; $_{R:}$ *Rotational*
Kinetic Energy	$KE_L = \frac{1}{2} mv^2$	$KE_R = \frac{1}{2} I \omega^2$	
Work-KE Theorem	$\sum W_L = \Delta KE_L$	$\sum W_R = \Delta KE_R$	
Mechanical Energy	$ME = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2 + mgh + \frac{1}{2} k x^2$		
Power	$P = Fv$	$P = \tau \omega$	
Impulse	$J = Ft$	$J = \tau t$	
Momentum	$L = mv$	$L = I\omega$	
Impulse-Momentum Theorem	$Ft = mv_f - mv_i$	$\tau t = I\omega_f - I\omega_i$	
Momentum Conservation	$m_1v_1 + m_2v_2$ $= m_1v_{1f} + m_2v_{2f}$	$I_1\omega_1 + I_2\omega_2$ $= I_1\omega_{1f} + I_2\omega_{2f}$	∑W=0: linear momentum conservation ∑τ=0: Angular momentum conservation

4. Parallel Axis Theorem

The moment of inertia of any object about an axis through its center of mass is the minimum moment of inertia for an axis in that direction in space. The moment of inertia about any axis parallel to that axis through the center of mass is given by

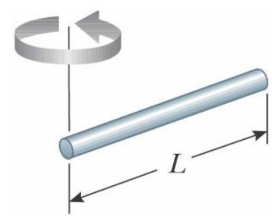
$$I_{\text{parallel axis}} = I_{\text{cm}} + Md^2$$



The expression added to the center of mass moment of inertia will be recognized as the moment of inertia of a point mass - the moment of inertia about a parallel axis is the center of mass moment plus the moment of inertia of the entire object treated as a point mass at the center of mass.

Example

1. What's the rotational inertia of a thin rod rotating one end axis?



Answer:

The moment of inertia of the rod about its centre is

$$I_{CM} = \frac{1}{12}ML^2$$

D is ½ *L*

Therefore,

$$I = I_{CM} + MD^2$$

$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$

5. Perpendicular Axis Theorem

The perpendicular axis theorem for planar objects can be demonstrated by looking at the contribution to the three axis moments of inertia from an arbitrary mass element. From the point mass moment, the contributions to each of the axis moments of inertia are

$$\Delta I_x = \Delta m y^2 \quad \Delta I_y = \Delta m x^2 \quad \Delta I_z = \Delta m r^2$$

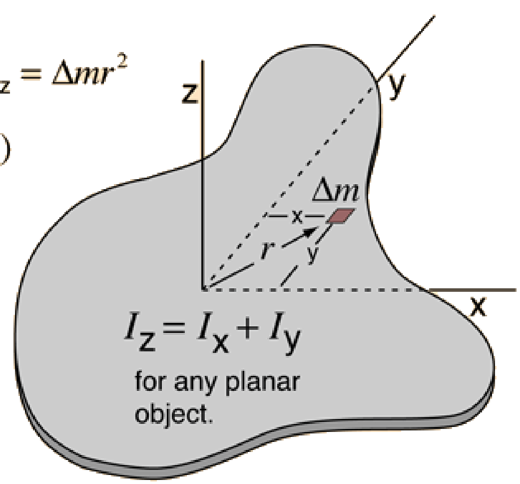
Then $\Delta I_x + \Delta I_y = \Delta m(x^2 + y^2)$

but since $r^2 = x^2 + y^2$

it follows that

$$\Delta I_x + \Delta I_y = \Delta m r^2 = \Delta I_z$$

Since this is true for any mass element then $I_x + I_y = I_z$



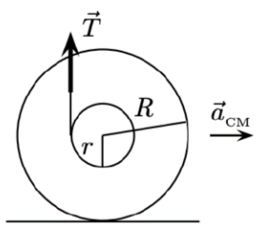
Practice

1.

In a standard physics lecture demonstration, a spool with inner radius r and outer radius R is pulled by a string wrapped tightly around the inner part of the spool. The string is pulled in such a way that the upward force remains constant as the spool rolls. In this case, the spool will roll to the right.

If it rolls without slipping on a horizontal tabletop, has mass M and a rotational inertia of $\mathcal{I}_{CM}$ about its center of mass, what is the acceleration of its center of mass (as a simple fraction)?

(Ignore friction)



1. $a_{CM} = \dfrac{T}{M R^2}$

2. $a_{CM} = \dfrac{T\,r\,R}{\mathcal{I}_{CM} + M\,r^2}$

3. $a_{CM} = \dfrac{T\,r\,R}{\mathcal{I}_{CM}}$

4. $a_{CM} = \dfrac{T\,r\,R}{\mathcal{I}_{CM} + M\,R^2}$

5. $a_{CM} = \dfrac{T\,r}{R\,\mathcal{I}_{CM}}$

6. $a_{CM} = \dfrac{T\,R^3}{M\,R^2\,r + \mathcal{I}_{CM}\,r}$

7. $a_{CM} = \dfrac{T}{M + \mathcal{I}_{CM}}$

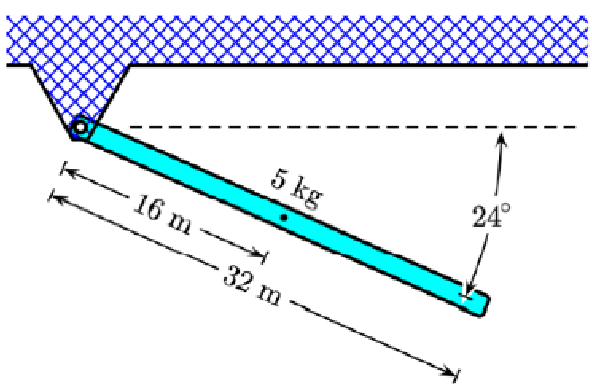
8. $a_{CM} = \dfrac{T\,r}{R\,M}$

9. $a_{CM} = \dfrac{T}{M}$

10. $a_{CM} = \dfrac{T\,R^2}{\mathcal{I}_{CM}}$

2.

A uniform 5 kg rod with length 32 m has a frictionless pivot at one end. The rod is released from rest at an angle of 24° beneath the horizontal.

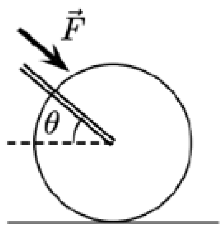


What is the angular acceleration of the rod immediately after it is released? The moment of inertia of a rod about the center of mass is $\frac{1}{12}\,m\,L^2$, where m is the mass of the rod and L is the length of the rod. The moment of inertia of a rod about either end is $\frac{1}{3}\,m\,L^2$, and the acceleration of gravity is $9.8\ \text{m/s}^2$.

Answer in units of rad/s^2.

3.

A lawn roller of mass M, radius R and rotational inertia $I_{\mathrm{CM}} = \kappa\, M\, R^2$ about its center of mass is being pushed across a level lawn by a constant force $\vec{F}$ applied parallel to the handle, which makes an angle θ with the horizontal. The roller rolls without slipping.



(a)

What is the acceleration of the lawn roller?

1. $a = \dfrac{F\ \cos\theta}{(\kappa+1)\, M}$
2. $a = \dfrac{\kappa\, F\ \cos\theta}{M}$
3. $a = \dfrac{F\ \cos\theta}{\kappa\, M}$
4. $a = \dfrac{F\ \cos\theta}{M}$
5. $a = \dfrac{F}{M}$
6. None of these
7. $a = \dfrac{\kappa\, F\ \cos\theta}{(\kappa+1)\, M}$
8. $a = \dfrac{(\kappa-1)\, F\ \cos\theta}{M}$

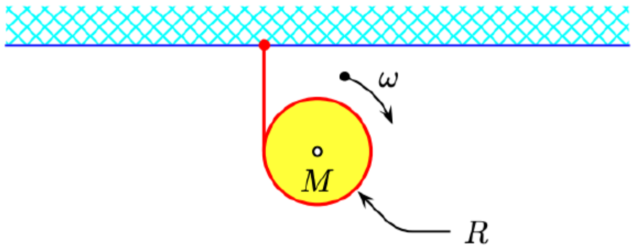
(b)

What is the force of static friction acting on the lawn roller as it accelerates?

1. $f_s = \dfrac{F\ \cos\theta}{\kappa}$
2. $f_s = \dfrac{\kappa\, F\ \cos\theta}{\kappa+1}$
3. $f_s = (\kappa-1)\, F\ \cos\theta$
4. $f_s = \kappa\, F \cos\theta$
5. None of these
6. $f_s = \kappa\, F\ \cos\theta$
7. $f_s = F\ \cos\theta$
8. $f_s = \dfrac{F\ \cos\theta}{\kappa+1}$

4.

A massless rope is wrapped around a uniform cylinder of radius R and mass M.



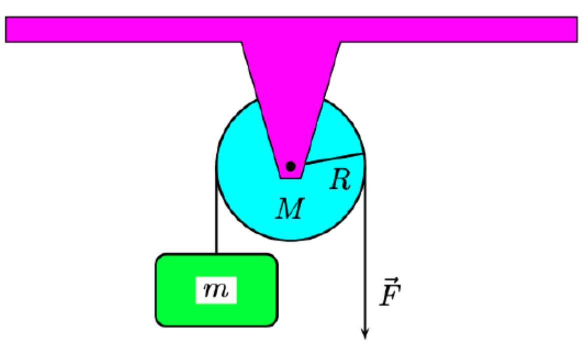
What is the linear acceleration of the cylinder? Assume the unwrapped portion of the rope is vertical and the axis of the cylinder remains horizontal.

(The moment of this solid cylinder about its center axis is $\frac{1}{2}MR^2$)

1. $a = \frac{3g}{4}$
2. $a = \frac{g}{2}$
3. $a = \frac{2g}{3}$
4. $a = \frac{3g}{5}$
5. $a = \frac{2g}{5}$
6. $a = \frac{g}{3}$
7. $a = g$
8. $a = 2g$
9. $a = \frac{g}{5}$
10. $a = \frac{g}{4}$

5.

A pulley of mass M and radius R pivots freely about its center with $\mathcal{I}_{cm} = \kappa M R^2$, where κ is some numerical constant. A string is attached to a mass m and run over the pulley as in the sketch.



If a downward force $\vec{F}$ is applied by your hand to the string, find the acceleration of the system in terms of F, m, M, κ and g only.

1. $a = \frac{F + mg}{\kappa M + m}$
2. $a = \frac{F - mg}{\kappa M + m}$
3. $a = \frac{F - mg}{\kappa M - m}$
4. $a = \frac{F + mg}{\kappa M - m}$
5. $a = \frac{F - mg}{(\kappa + 1) M}$
6. $a = \frac{F - mg}{(\kappa + 1) m}$

Answer:

1. (3)
2. 0.42 rad/s^2
3. (a): (1); (b):(2)
4. (3)
5. (2)